Solutions to Question Sheet 2, Limits II. v1. 2019-20

Limits not existing

In the next two questions we look at examples of functions that do **not** have a limit at a point.

1. *Recall the unproved result from lectures that if you can find two sequences*

   $\{x_n\}_{n\geq 1}$, $x_n \neq a$ *for all* $n \geq 1, \lim_{n\to\infty} x_n = a$ *and*

   $\{y_n\}_{n\geq 1}$, $y_n \neq a$ *for all* $n \geq 1, \lim_{n\to\infty} y_n = a$

   for which $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$ *then* $\lim_{x\to a} f(x)$ *does not exist.*

   Use this to prove that

   i) For $f_1 : \mathbb{R} \to \mathbb{R}$ defined by

$$f_1(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational,} \end{cases}$$

   the limit $\lim_{x\to 0} f_1(x)$ does not exist.

   Hint find a sequence $x_n \to 0$ for which $f_1(x_n) = 1$ for all $n \geq 1$ and a sequence $y_n \to 0$ with $f_1(y_n) = 0$ for all $n \geq 1$.

   ii) For $f_2 : \mathbb{R}\setminus\{0\} \to \mathbb{R}$ defined by

$$f_2(x) = \sin\left(\frac{\pi}{x}\right),$$

   the limit $\lim_{x\to 0} f_2(x)$ does not exist.

   Hint find a sequence $x_n \to 0$ for which $\sin(\pi/x_n) = 1$ for all $n \geq 1$ and a sequence $y_n \to 0$ with $\sin(\pi/y_n) = -1$ for all $n \geq 1$.

   Solution i) First we need a sequence of non-zero *rational* numbers tending to 0. Let

$$x_n = \frac{1}{n}$$

for all $n \geq 1$. Then $\lim_{n\to\infty} x_n = 0$. Since each x_n is rational we have $f_1(x_n) = 1$ for all $n \geq 1$ in which case $\lim_{n\to\infty} f_1(x_n) = 1$.

Next we need a sequence of non-zero *irrational* numbers tending to 0. Let

$$y_n = \frac{\sqrt{2}}{n}$$

for all $n \geq 1$. Then $\lim_{n\to\infty} y_n = 0$. This time each y_n is irrational so $f_1(y_n) = 0$ for all $n \geq 1$ in which case $\lim_{n\to\infty} f_1(y_n) = 0$.

By the result quoted in the question we deduce that $\lim_{x\to 0} f_1(x)$ does not exist.

ii) We know that

$$\sin\left(2\pi n + \frac{\pi}{2}\right) = 1$$

for all integers n, so choose x_n such that

$$\frac{\pi}{x_n} = 2\pi n + \frac{\pi}{2}, \quad \text{i.e.} \quad x_n = \frac{2}{4n+1}.$$

for all $n \geq 1$. Then $\lim_{n\to\infty} x_n = 0$ while $f_2(x_n) = 1$ for all $n \geq 1$ so $\lim_{n\to\infty} f_2(x_n) = 1$.

We also know that

$$\sin\left(2\pi n + \frac{3\pi}{2}\right) = -1$$

for all integers n, so choose y_n such that

$$\frac{\pi}{y_n} = 2\pi n + \frac{3\pi}{2}, \quad \text{i.e.} \quad y_n = \frac{2}{4n+3}.$$

for all $n \geq 1$. Then $\lim_{n\to\infty} y_n = 0$ while $f_2(y_n) = -1$ for all $n \geq 1$ so $\lim_{n\to\infty} f_2(y_n) = -1$.

Thus again by the result quoted in the question we deduce that $\lim_{x\to 0} f_2(x)$ does not exist.

2. i) Show by means of an example that $\lim_{x\to a} \{f(x) + g(x)\}$ may exist even though neither $\lim_{x\to a} f(x)$ or $\lim_{x\to a} g(x)$ exist.

   ii) Do the same for $\lim_{x\to a} f(x)\, g(x)$.

   Hint: Construct f and g from a function in the previous question or a similar one seen in the notes.

   Solution The hint suggests using a function that fails to have a limit at a point. From the notes we have that $h(x) = x/|x|$ $x \neq 0$, with $a = 0$, is such a function. (Recall, the left hand limit does not equal the right hand limit at 0).

   i) For example, with $a = 0$, try

$$f(x) = \frac{x}{|x|} \qquad \text{and} \qquad g(x) = -\frac{x}{|x|}$$

   for $x \neq 0$.

   ii) For example, with $a = 0$, try

$$f(x) = \frac{x}{|x|} \qquad \text{and} \qquad g(x) = \frac{x}{|x|}$$

   for $x \neq 0$. In this case

$$\lim_{x\to 0} f(x)\, g(x) = \lim_{x\to 0} \frac{x^2}{|x|^2} = \lim_{x\to 0} 1 = 1$$

   exists.

One-sided limits

In the next two questions we examine a limit at a point by examining the two one-sided limits at that point.

3. Let

$$F(x) = \frac{x^2 - 1}{|x - 1|}.$$

   i) For what x is this well-defined?

Hint Recall that $|y| = y$ if $y \geq 0$ and $= -y$ if $y < 0$.

ii) Find $\lim_{x\to 1^+} F(x)$

iii) Find $\lim_{x\to 1^-} F(x)$.

iv) Does $\lim_{x\to 1} F(x)$ exist?

v) Sketch the graph of $F(x)\,, x \in \mathbb{R}$.

Solution i) $F(x)$ is well-defined for $x \neq 1$.

ii) If $x > 1$ then $x - 1 > 0$ and so $|x-1| = x - 1$ and thus

$$F(x) = \frac{x^2-1}{x-1} = x + 1.$$

Hence $\lim_{x\to 1+} F(x) = 2$.

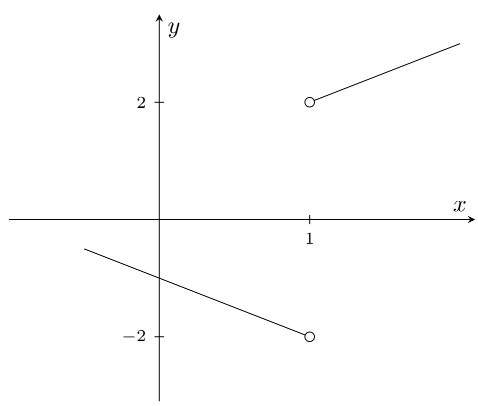
iii) If $x < 1$ then $x - 1 < 0$ and so $|x-1| = -(x-1)$ and thus

$$F(x) = \frac{x^2-1}{-(x-1)} = -(x+1)\,.$$

Hence $\lim_{x\to 1-} F(x) = -2$.

iv) The one-sided limits are not equal and thus $\lim_{x\to 1} F(x)$ does **not** exist.

v) The graph of F :



4. Sketch the graph of

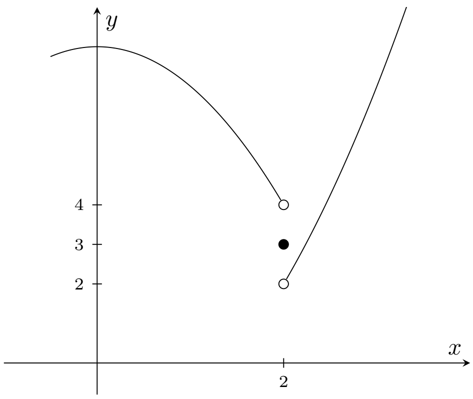
$$f(x) = \begin{cases} 8 - x^2, & x < 2 \\ 3, & x = 2 \\ x^2 - 2 & x > 2. \end{cases}$$

Use the ε - δ definition to evaluate the following one-sided limits.

$$\text{i)} \lim_{x \to 2-} f(x) \qquad \text{and} \qquad \text{ii)} \lim_{x \to 2+} f(x) \,.$$

Does $\lim_{x \to 2} f(x)$ exist?

Solution The graph of f is



i) *Left hand limit*: From the graph we might guess that $\lim_{x \to 2-} f(x) = 4$.

For $x < 2$ we have

$$|f(x) - 4| = \left|\left(8 - x^2\right) - 4\right| = \left|4 - x^2\right| = |x - 2|\,|x + 2| \,. \qquad (1)$$

Let $\varepsilon > 0$ be given, choose $\delta = \min(1, \varepsilon/4)$ and assume $2 - \delta < x < 2$.

The bound $\delta \leq 1$ means that $1 < x < 2$, and so $3 < x + 2 < 4$. Thus $|x + 2| < 4$ and (1) becomes $|f(x) - 4| < 4\,|x - 2|$.

The other bound $\delta \leq \varepsilon/4$ then means that

$$|f(x) - 4| < 4\,|x - 2| < 4\delta \leq 4\left(\frac{\varepsilon}{4}\right) = \varepsilon.$$

Thus we have verified the ε - δ definition of the *one-sided* limit $\lim_{x\to 2-} f(x) = 4$.

ii) *Right hand limit*: From the graph we might guess that $\lim_{x\to 2+} f(x) = 2$.

For $x > 2$ we have $f(x) = x^2 - 2$ and so

$$|f(x) - 2| = \left|x^2 - 4\right| = |x-2|\,|x+2|\,. \tag{2}$$

Let $\varepsilon > 0$ be given, choose $\delta = \min(1, \varepsilon/5)$ and assume $2 < x < 2 + \delta$.

The bound $\delta \leq 1$ means that $2 < x < 3$, and so $4 < x + 2 < 5$. Thus $|x+2| < 5$ and (2) becomes $|f(x) - 4| < 5\,|x-2|$.

The other bound, $\delta \leq \varepsilon/5$, means that

$$|f(x) - 2| < 5\,|x-2| < 5\delta \leq 5\left(\frac{\varepsilon}{5}\right) = \varepsilon.$$

Thus we have verified the ε - δ definition of the *one-sided* limit $\lim_{x\to 2+} f(x) = 2$.

Since the one-sided limits do **not** agree we conclude that $\lim_{x\to 2} f(x)$ does **not** exist.

Limits at Infinity

In the next five questions we look at limits as $x \to +\infty$ and $x \to -\infty$.

5. Verify the ε - X definition of

$$\lim_{x\to+\infty} \frac{3x+3}{x-2} = 3.$$

Solution *Rough work* Consider

$$|f(x) - L| = \left|\frac{3x+3}{x-2} - 3\right| = \frac{9}{|x-2|}.$$

If $x > 2$ then $|x-2| = x-2$ and we demand that

$$\frac{9}{x-2} < \varepsilon, \quad \text{i.e.} \quad x > 2 + \frac{9}{\varepsilon}.$$

If x satisfies this then necessarily $x > 2$ as required for the previous step. *End of Rough work*

Proof Let $\varepsilon > 0$ be given, choose $X = 2 + 9/\varepsilon$ and assume $x > X$. For such x we have

$$\begin{aligned} |f(x) - L| &= \left| \frac{3x+3}{x-2} - 3 \right| = \frac{9}{|x-2|} \\ &= \frac{9}{x-2} \quad \text{since } x > X > 2 \\ &< \frac{9}{X-2} \quad \text{since } x > X \\ &= \frac{9}{(9/\varepsilon)} \quad \text{since } X - 2 = 9/\varepsilon \\ &= \varepsilon. \end{aligned}$$

Therefore we have verified the ε - X definition of

$$\lim_{x \to +\infty} \frac{3x+3}{x-2} = 3.$$

6. Verify the ε - X definition of

$$\lim_{x \to -\infty} \frac{2x-2}{x+2} = 2.$$

Solution *Rough Work.* Consider

$$|f(x) - L| = \left| \frac{2x-2}{x+2} - 2 \right| = \frac{6}{|x+2|}.$$

In the definition of limit as $x \to -\infty$ we need to find a **negative** X. First demand that $x < -2$ so $x + 2 < 0$ and $|x+2| = -(x+2)$. Then

$$\left| \frac{2x-2}{x+2} - 2 \right| = -\frac{6}{x+2}.$$

Next demand this is $< \varepsilon$, that is

$$-\frac{6}{x+2} < \varepsilon.$$

This can rearranged as $x < -2 - 6/\varepsilon$. If x satisfies this then necessarily $x < -2$ as previously demanded.
textitEnd of Rough work.

Proof Let $\varepsilon > 0$ be given. Choose $X = -2 - 6/\varepsilon$. Assume $x < X$. First $x < X < -2$ means $x + 2$ is negative so $|x+2| = -(x+2)$ and thus

$$|f(x) - L| = \left|\frac{2x-2}{x+2} - 2\right| = \frac{6}{|x+2|} = -\frac{6}{(x+2)}.$$

But $x < X = -2 - 6/\varepsilon$ implies $-(x+2) > 6/\varepsilon$, i.e.

$$-\frac{6}{(x+2)} < \varepsilon.$$

Hence $|f(x) - L| < \varepsilon$ and we have verified the definition of

$$\lim_{x\to-\infty} \frac{2x-2}{x+2} = 2.$$

7. Find the value of

$$\lim_{x\to+\infty} \frac{2-x^2}{x^2+2}$$

and show your value satisfies the ε - X definition.

Solution *Rough work* For large x, whatever it's sign,

$$\frac{2-x^2}{x^2+2} \quad \text{'looks like'} \quad \frac{-x^2}{x^2} = -1.$$

So we guess the limit is -1. Consider

$$|f(x) - L| = \left|\frac{2-x^2}{x^2+2} - (-1)\right| = \frac{4}{|x^2+2|} = \frac{4}{x^2+2}.$$

You could demand this is $< \varepsilon$ i.e.

$$x > \left(-2 + \frac{4}{\varepsilon}\right)^{1/2}.$$

But this requires $-2 + 4/\varepsilon > 0$, i.e. $\varepsilon < 2$ and we don't usually restrict ε. Instead, bound $|f(x) - L|$ from above by a 'simpler' function and then demand the bound is $< \varepsilon$. For example,

$$\frac{4}{x^2 + 2} < \frac{4}{x^2}$$

so demand $4/x^2 \le \varepsilon$. And if you don't like square roots of ε, require $x > 1$ which implies $x^2 > x$ and thus $4/x^2 \le 4/x$. We need only then demand $4/x < \varepsilon$.

End of Rough work

Proof Let $\varepsilon > 0$ be given, choose $X = \max(1, 4/\varepsilon)$ and assume $x > X$. For such x we have

$$\begin{aligned} |f(x) - L| &= \frac{4}{x^2 + 2} < \frac{4}{x^2} < \frac{4}{x} \quad \text{since } x > X \ge 1 \\ &\le \frac{4}{(4/\varepsilon)} \quad \text{since } x > X \ge 4/\varepsilon \\ &= \varepsilon. \end{aligned}$$

Therefore we have verified the ε - X definition of

$$\lim_{x \to +\infty} \frac{2 - x^2}{x^2 + 2} = -1.$$

8. Find the value of

$$\lim_{x \to -\infty} \frac{3x + 3}{x - 2},$$

and show your value satisfies the ε - X definition.

Solution *Rough Work* For large x, whatever it's sign,

$$\frac{3x + 3}{x - 2} \quad \text{'looks like'} \quad \frac{3x}{x} = 3.$$

So we guess the limit is -1. Consider

$$|f(x) - L| = \left|\frac{3x+3}{x-2} - 3\right| = \frac{9}{|x-2|}.$$

In the definition of limit as $x \to -\infty$ we need to find a **negative** X. If $x < X < 0$ then $x - 2 < 0$ so $|x-2| = -(x-2)$. We need then demand that

$$\frac{9}{2-x} < \varepsilon.$$

End of Rough Work

Proof Let $\varepsilon > 0$ be given, choose $X = \min(0, 2 - 9/\varepsilon)$ and assume $x < X$. For such x we have

$$\begin{aligned} |f(x) - L| &= \frac{9}{|x-2|} = \frac{9}{2-x} \quad \text{since } x < X \leq 0 \\ &\leq \varepsilon \quad \text{since } x < X \leq 2 - 9/\varepsilon. \end{aligned}$$

Therefore we have verified the ε-X definition of

$$\lim_{x\to-\infty} \frac{3x+3}{x-2} = 3.$$

Note $x - 2 < 0$ will follow from $x < 2$ which, in turn, follows from $x < 2 - 9/\varepsilon$. Thus you might be tempted to choose $X = 2 - 9/\varepsilon$. Yet, as mentioned in the rough work we look for *negative* X and $2 - 9/\varepsilon < 0$ only if $\varepsilon < 9/2$. As stated before we don't restrict ε, hence our choice of $X = \min(0, 2 - 9/\varepsilon)$. **End of Note**

Extra questions for practice

9. Verify the ε-X definition of

$$\lim_{x\to+\infty} \frac{-2-x^2}{x^2-2} = -1.$$

Solution *Rough work* Consider

$$|f(x) - L| = \frac{4}{|x^2-2|}.$$

If we assume $x^2 > 2$ then $|f(x) - L| = 4/(x^2 - 2)$ and we need demand this is $< \varepsilon$. This rearranges as $x > \sqrt{2 + 4/\varepsilon}$. If this holds then the previous requirement $x > \sqrt{2}$ necessarily holds.

End of Rough work.

Proof Let $\varepsilon > 0$ be given, choose $X = \sqrt{2 + 4/\varepsilon}$ and assume $x > X$. For such x we have

$$\begin{aligned} |f(x) - L| &= \frac{4}{|x^2 - 2|} = \frac{4}{x^2 - 2} \quad \text{since } x > X \geq \sqrt{2} \\ &< \frac{4}{X^2 - 2} \quad \text{since } x > X \\ &= \varepsilon \end{aligned}$$

by the definition of X. Therefore we have verified the ε-X definition of

$$\lim_{x \to +\infty} \frac{-2 - x^2}{x^2 + 2} = -1.$$

Note it is not so easy to use the method of the previous question, and bound $|f(x) - L|$ from above by a simpler function. For example it is **not** true to say

$$\frac{4}{x^2 - 2} \leq \frac{4}{x^2}.$$

It is, though, true that

$$\frac{4}{x^2 - 2} \leq \frac{5}{x^2}$$

for $x^2 \geq 10$, or, if you don't like square roots, $x \geq 4$. I have chosen 5 on the right hand side only since it is the smallest integer strictly larger than 4. If $x \geq 4$ then $5/x^2 < 5/4x$ and we need demand this is $< \varepsilon$.

Alternative proof Let $\varepsilon > 0$ be given, choose $X = \max(4, 5/4\varepsilon)$ and assume $x > X$. For such x we have

$$\begin{aligned}
|f(x) - L| &= \frac{4}{x^2 - 2} \quad \text{since } x \geq X \geq 4 \implies x^2 - 2 > 0 \\
&< \frac{5}{x^2} \quad \text{since } x \geq X \geq 4 > \sqrt{10} \\
&< \frac{5}{4x} \quad \text{since } x \geq X \geq 4 \\
&\leq \frac{5}{4X} \leq \frac{5}{4\,(5/4\varepsilon)} = \varepsilon \quad \text{since } x \geq X \geq 5/4\varepsilon.
\end{aligned}$$

End of Note

10. Find the value of

$$\lim_{x \to -\infty} \frac{-2 - x^2}{x^2 - 2},$$

and show your value satisfies the ε - X definition.

Solution For large x

$$\frac{-2 - x^2}{x^2 - 2} \text{ `looks like' } \frac{-x^2}{x^2} = -1,$$

so we guess the limit is -1. Let $\varepsilon > 0$ be given, choose $X = -\max(4, 5/4\varepsilon) = \min(-4, -5/4\varepsilon)$ and assume $x < X$. For such x we have

$$\begin{aligned}
|f(x) - L| &= \frac{4}{x^2 - 2} \quad \text{since } x \leq X \leq -4 \implies x^2 - 2 > 0 \\
&< \frac{5}{x^2} \quad \text{since } x \leq -4 \implies x^2 \geq 10 \\
&< \frac{5}{4\,|x|} \quad \text{since } x \leq -4 \implies x^2 \geq 4\,|x| \\
&\leq \frac{5}{4\,|X|} \leq \frac{5}{4\,(5/4\varepsilon)} = \varepsilon \quad \text{since } |x| \geq |X| \geq 5/4\varepsilon.
\end{aligned}$$

Hence we have verified the ε - X definition of

$$\lim_{x \to -\infty} \frac{-2 - x^2}{x^2 - 2} = -1.$$

Aside I chose 4 in $-\max(4, 5/(4\varepsilon))$ simply as the smallest integer satisfying $x^2 > 10$. You could choose $\sqrt{10}$ in its place which would lead to the choice of $X = -\max\left(\sqrt{10}, \sqrt{5}/\left(\sqrt{2}\varepsilon\right)\right)$.